Indian Statistical Institute Bangalore
Statistics and Mathematics Unit
M. Math II Year
**Course: Several Complex Variables**
Final Exam, Date: 23 April, 2025

Time: 3 hours | Maximum marks: 50

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**All questions are compulsory.**

NOTATIONS: (1) $\mathbb{R}$: set of all real numbers, (2) $\mathbb{C}$: set of all complex numbers, (3) $\mathbb{C}^n$: the $n$-dimensional complex number space, (4) $\mathbb{R}^n$: the $n$-dimensional real number space, (5) For a given $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n$ and $\mathbf{r} = (r_1, r_2, \dots, r_n) \in \mathbb{R}^n$ with $r_i > 0$ for all $1 \leq i \leq n$,

$$\mathbb{D}^n(\mathbf{a}, \mathbf{r}) := \{\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n : |z_i - a_i| < r_i \text{ for all } 1 \leq i \leq n\}.$$

For $n = 1, a \in \mathbb{C}$ and a positive $r \in \mathbb{R}$, we will write $\mathbb{D}(a, r)$, instead of $\mathbb{D}^1(\mathbf{a}, \mathbf{r})$.

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1. Let $f : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic in all of $\mathbb{C}^2$, and

$$\int_{\theta_1=0}^{2\pi} \int_{\theta_2=0}^{2\pi} \left| f(re^{i\theta_1}, re^{i\theta_2}) \right| d\theta_2 d\theta_1 \leq r^{\frac{17}{3}}$$

for all real numbers $r > 0$. Prove that $f$ is identically 0. **(8 marks)**

2. Suppose $f$ is a complex-valued holomorphic function defined in

$$\Omega = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 2, |z_2| < \frac{1}{2} \right\} \bigcup \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < \frac{1}{2}, |z_2| < 2 \right\}.$$

Show that $f$ has an analytic continuation to the domain

$$\Omega' = \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 2, |z_2| < 2, |z_1 z_2| < 1 \right\}.$$

**(8 marks)**

3. (a) Write the definitions for domain of holomorphy and holomorphically convex domain in $\mathbb{C}^n$. What is the relation between them?

**(2 marks + 2 marks)**

(b) Prove that any convex domain $\Omega \subset \mathbb{C}^n$ is a domain of holomorphy (you may assume without proof the supporting hyperplane theorem).

**(7 marks)**

4. (a) Let $\Omega \subset \mathbb{C}^n$ be a bounded connected domain and $\mathbf{0} \in \Omega$. Also let $f : \Omega \to \Omega$ be a holomorphic map such that $f(\mathbf{0}) = \mathbf{0}$ and the Jacobian $J_f(\mathbf{0})$ is equal to the $n \times n$ identity matrix. Prove that

$$f(\mathbf{z}) = \mathbf{z}$$

for all $\mathbf{z} \in \Omega$. (Here $\mathbf{z} = (z_1, z_2, \dots, z_n)$ and $\mathbf{0} = (0, 0, \dots, 0)$ are elements of $\mathbb{C}^n$).

**(8 marks)**

(b) Show that $f : B \to H$ defined by

$$f(z_1, z_2) = \left( \frac{z_1}{1+z_2}, i\frac{1-z_2}{1+z_2} \right)$$

is a $1-1$ holomorphic map of

$$B = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

onto

$$H = \{(z_1, z_2) \in \mathbb{C}^2 : \Im(z_2) > |z_1|^2\}.$$

(Here $\Im(w)$ denotes the imaginary part of a $w \in \mathbb{C}$).

**(7 marks)**

5. Suppose $f$ is a complex-valued holomorphic function defined on $\mathbb{D}^n(\mathbf{0}, \mathbf{r})$, $n \geq 2$. Also suppose that $f$ vanishes exactly of order $k$ relative to $z_n$ at the origin. Show that there exists a (smaller) polydisc $\mathbb{D}^n(\mathbf{0}, \mathbf{R})$:

$$\mathbf{R} = (\mathbf{R}', R_n), \text{ where } \mathbf{R}' = (R_1, R_2, \ldots, R_{n-1}) \in \mathbb{R}^{n-1},$$

and a number $\epsilon > 0$ such that

$$f(\mathbf{0}', z_n) \neq 0 \text{ for } 0 < |z_n| \leq R_n,$$

$$f(\mathbf{z}', z_n) \neq 0 \text{ for } \mathbf{z}' \in \mathbb{D}^{n-1}(\mathbf{0}', \mathbf{R}'), \; R_n - \epsilon < |z_n| < R_n + \epsilon,$$

where $\mathbf{0}' = (0, 0, \ldots, 0) \in \mathbb{C}^{n-1}$. Moreover, show that for any $\mathbf{z}' \in \mathbb{D}^{n-1}(\mathbf{0}', \mathbf{R}')$, the function $g(z_n) = f(\mathbf{z}', z_n)$ will have precisely $k$ zeroes (counting multiplicities) in the disc $\mathbb{D}(0, R_n)$.

**(8 marks)**

## Best wishes!